Derivation of the Transfer Function of the Moog Ladder Filter

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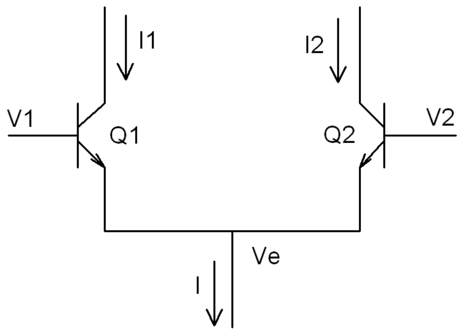
Much of this note draws heavily on the standard analysis of the basic differential or long-tailed pair. I find the analysis that utilizes the hyperbolic tangent to be particularly succinct and elegant—the following is based around that to be found in [6], but I have added some of the missing detail here. Suppose we have a pair of NPN transistors tied at the emitter, so the common emitter voltage is V_E, and from which point we draw current I. The base voltages are V_1 and V_2, and the collector currents are I_1 and I_2, which we also assume are the emitter currents (i.e. neglect the base currents):



With I_s the saturation current as normal, and writing kT/q as V_T for convenience, we have

$$I_1 \approx I_s e^{\frac{V_1 - V_E}{V_T}} \quad \text{and} \quad I_2 \approx I_s e^{\frac{V_2 - V_E}{V_T}}.$$

Divide to get

$$\frac{I_2}{I_1} = \frac{e^{\frac{V_2 - V_E}{V_T}}}{e^{\frac{V_1 - V_E}{V_T}}} = e^{\frac{V_2 - V_1}{V_T}}. \tag{1}$$

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A good place to seek answers to questions on the internals of synthesizers in general is the 'Synth DIY' mailing list: `http://www.euronet.nl/~rja/Emusic/Synth-diy/`

Substitute for I_2 in

$$I = I_1 + I_2 = I_1 + I_1 e^{\frac{V_2 - V_1}{V_T}}, \tag{2}$$

giving

$$I_1 = \frac{I}{1 + e^{\frac{V_2 - V_1}{V_T}}}.$$

Now the subtle bit: multiply top and bottom by 2, and 'add zero' into the numerator:

$$I_1 = \frac{I \times 2}{2\left(1 + e^{\frac{V_2 - V_1}{V_T}}\right)} = \frac{I}{2}\left[\frac{1 + 1 + e^{\frac{V_2 - V_1}{V_T}} - e^{\frac{V_2 - V_1}{V_T}}}{1 + e^{\frac{V_2 - V_1}{V_T}}}\right] = \frac{I}{2}\left[1 + \frac{1 - e^{\frac{V_2 - V_1}{V_T}}}{1 + e^{\frac{V_2 - V_1}{V_T}}}\right].$$

Now

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1},$$

from which it can be seen we get

$$I_1 = \frac{I}{2}\left[1 - \tanh\left(\frac{V_2 - V_1}{2V_T}\right)\right] = \frac{I}{2}\left[1 + \tanh\left(\frac{V_1 - V_2}{2V_T}\right)\right]. \tag{3}$$

For I_2 we have

$$I_2 = I - I_1 = I - \frac{I}{2}\left[1 + \tanh\left(\frac{V_1 - V_2}{2V_T}\right)\right] = \frac{I}{2}\left[1 - \tanh\left(\frac{V_1 - V_2}{2V_T}\right)\right],$$

and the symmetry between the currents is very obvious.

For the linear approximation when $(V_1 - V_2)/2V_T$ is small, the Taylor series expansion of $\tanh x$ is $x - \frac{x^3}{3} + \cdots$, so when the differential input voltage is small, say within ± 20mV, we can aproximate as

$$I_{1,2} = \frac{I}{2}\left[1 \pm \frac{V_1 - V_2}{2V_T}\right], \tag{4}$$

with 'plus' for I_1, 'minus' for I_2. (Note that with $V_T = 0.026$V, the constant $1/(2V_T) = 19.2$, one that often crops up on OTA datasheets.)

Now from equations (1) and (2) it is clear that when the emitter voltages are not equal, i.e. we have V_{E_1} and V_{E_2}, if the currents sum to give I, we will get a more general version of equation (3)

$$I_{1,2} = \frac{I}{2}\left[1 \pm \tanh\left(\frac{V_1 - V_2 - V_{E_1} + V_{E_2}}{2V_T}\right)\right], \tag{5}$$

and also its linear approximation

$$I_{1,2} = \frac{I}{2}\left[1 \pm \frac{V_1 - V_2 - V_{E_1} + V_{E_2}}{2V_T}\right]. \tag{6}$$

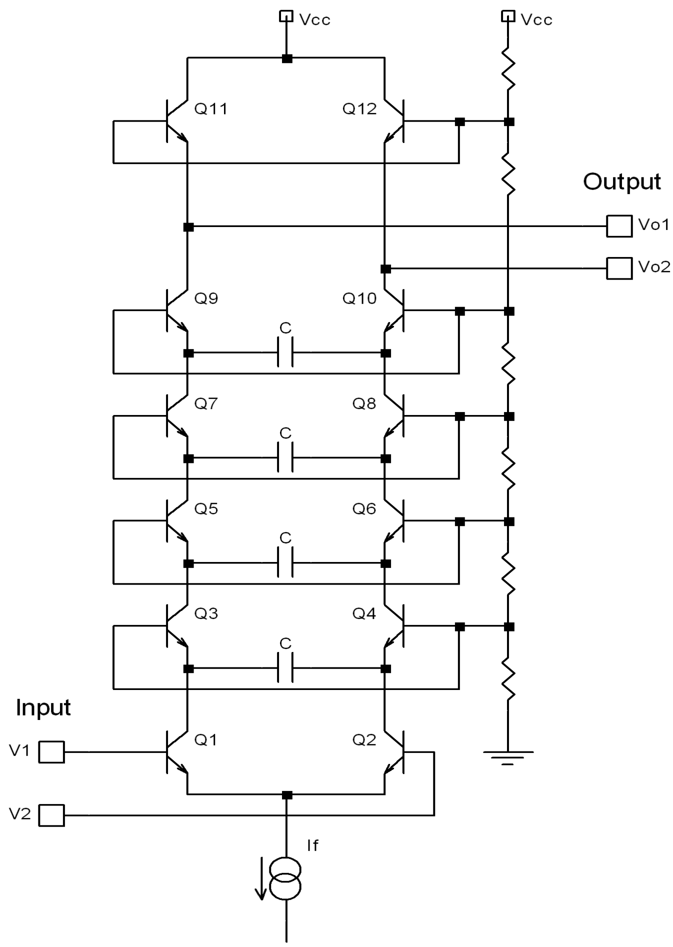

Figure 1: Basic circuit of the Moog ladder filter

Working with the basic filter set-up shown in Figure 1, as per Robert Moog's original patent [1], the filter consists of 'driver transistors' $Q1$ and $Q2$, to which the input voltages V_1 and V_2 are applied, and current I_f, proportional to the cut-off frequency, is drawn from the emitters; a pair of 'output coupling transistors' , $Q11$ and $Q12$, from which the output voltages V_{o1} and V_{o2} are taken; and in between, four filter stages, each consisting of a pair of transistors and a capacitor.

Throughout, assume that the base currents are negligible, and thus that through each transistor the emitter current equals the collector current.

Driver pair: let $V_{in} = V_1 - V_2$ be the input voltage, then with I_{C_1} and I_{C_2} the collector currents, we have $I_f = I_{C_1} + I_{C_2}$ and so from (4) we get

$$I_{C_1} = \frac{I_f}{2}\left[1 + \frac{V_{in}}{2V_T}\right] \quad \text{and} \quad I_{C_2} = \frac{I_f}{2}\left[1 - \frac{V_{in}}{2V_T}\right],$$

which on subtracting gives

$$I_{C_1} - I_{C_2} = \frac{I_f V_{in}}{2V_T}. \tag{7}$$

Output pair: let the output voltage be $V_{out} = V_{o1} - V_{o2} = V_{E_{11}} - V_{E_{12}}$; with collector currents $I_{C_{11}}$ and $I_{C_{12}}$, and assuming for now that $I_{C_{11}} + I_{C_{12}} = I_f$, from (6) we get

$$I_{C_{11}} = \frac{I_f}{2}\left[1 + \frac{-V_{E_{11}} + V_{E_{12}}}{2V_T}\right] = \frac{I_f}{2}\left[1 - \frac{V_{out}}{2V_T}\right] \quad \text{and} \quad I_{C_{12}} = \frac{I_f}{2}\left[1 + \frac{V_{out}}{2V_T}\right],$$

and again subtraction gives

$$I_{C_{11}} - I_{C_{12}} = -\frac{I_f V_{out}}{2V_T}. \tag{8}$$

Filter stage transistor pair: work with $Q3$ and $Q4$, but the working applies equally to the others. We now have capacitor C connected between the emitters: assume that current I flows from V_{E_3} to V_{E_4}, i.e. that we have

$$V_{E_3} - V_{E_4} = \frac{I}{sC}.$$

As usual, with collector currents I_{C_3} and I_{C_4}, we have

$$I_{C_3} = I_{C_1} + I \quad \text{and} \quad I_{C_4} + I = I_{C_2}.$$

If we add these two we see that

$$I_{C_3} + I_{C_4} = I_{C_1} + I_{C_2} = I_f,$$

which means that the sums of the collector currents all the way up the ladder equal I_f, and in particular, that this is so for the output pair, and so the assumption made above is correct. Subtracting the equations gives

$$I_{C_1} - I_{C_2} = I_{C_3} - I_{C_4} - 2I = I_{C_3} - I_{C_4} - 2sC(V_{E_3} - V_{E_4}).$$

Since $I_{C_3} + I_{C_4} = I_f$, we can again make use of (6) to get

$$I_{C_3} = \frac{I_f}{2}\left[1 + \frac{V_{E_4} - V_{E_3}}{2V_T}\right] \quad \text{and} \quad I_{C_4} = \frac{I_f}{2}\left[1 - \frac{V_{E_4} - V_{E_3}}{2V_T}\right],$$

which on subtracting give

$$I_{C_3} - I_{C_4} = \frac{I_f(V_{E_4} - V_{E_3})}{2V_T},$$

from which we usefully get

$$V_{E_4} - V_{E_3} = \frac{2V_T(I_{C_3} - I_{C_4})}{I_f}.$$

Rid this from the expression above to get

$$I_{C_1} - I_{C_2} = I_{C_3} - I_{C_4} + 2sC\left(\frac{2V_T(I_{C_3} - I_{C_4})}{I_f}\right),$$

from which it is a small step to get

$$\frac{I_{C_3} - I_{C_4}}{I_{C_1} - I_{C_2}} = \frac{1}{1 + s\dfrac{4V_TC}{I_f}}.$$

This is the transfer function of a single stage. If we put $R_{equiv} = 4V_T/I_f$, we could write this as

$$\frac{I_{C_3} - I_{C_4}}{I_{C_1} - I_{C_2}} = \frac{1}{1 + sCR_{equiv}},$$

or better still as

$$\frac{I_{C_3} - I_{C_4}}{I_{C_1} - I_{C_2}} = \frac{1}{1 + s/\omega_c},$$

where

$$\omega_c = \frac{1}{CR_{equiv}} = \frac{I_f}{4CV_T},$$

or

$$f_c = \frac{1}{2\pi CR_{equiv}} = \frac{I_f}{8\pi CV_T}.$$

It is clear that when the 4 stages are cascaded, we will get

$$\frac{I_{C_{11}} - I_{C_{12}}}{I_{C_1} - I_{C_2}} = \frac{1}{(1 + s/\omega_c)^4},$$

and using equations (8) and (7)

$$\frac{-\dfrac{I_fV_{out}}{2V_T}}{\dfrac{I_fV_{in}}{2V_T}} = \frac{1}{(1 + s/\omega_c)^4},$$

which after cancelling, finally gives what we are looking for

$$\frac{V_{out}}{V_{in}} = \frac{-1}{(1 + s/\omega_c)^4}.$$

Apart from the negation, this agrees with that given in [5], which is just about the only other detailed reference I know of on this filter. (The sign inversion is probably just due to the order chosen for the difference between V_{o1} and V_{o2}, and is likely to only be of consequence if considering feeding the output back to the input for resonance purposes.)

For $C = 10\text{nF}$, $V_T = 26\text{mV}$ and $I_f = 10\mu\text{A}$, we get $f_c = 1530\text{Hz}$, and at this frequency, $|V_{out}/V_{in}| = |1/(1+j)^4| = 1/\sqrt{(1+j)^4(1-j)^4} = 1/\sqrt{(1-j^2)^4} = 1/\sqrt{(1+1)^4} = 1/4 \equiv -12\text{dB}$. The output from a simple simulation, with biasing resistors to keep the

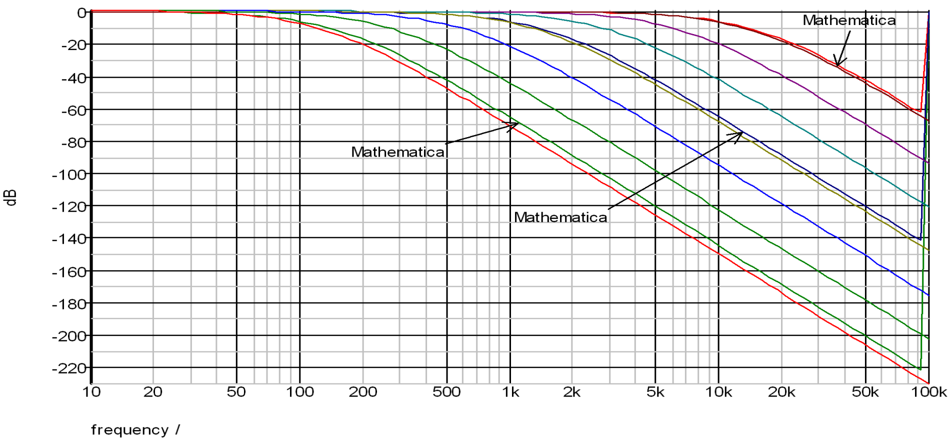

Figure 2: Frequency response - SIMetrix simulation versus Mathematica

transistors about 1V apart, is shown in Figure 2 - a multi-step run with I_f from 1μA to 100μA, 3 steps per decade. The middle trace is at $I_f = 10\mu$A, showing about -13dB at 1500Hz. Also on the graph are 3 traces from data generated in Mathematica using the above equations, at values of $I_f = 1\mu, 10\mu$ and 100μA - they show good agreement with the simulated data, confirming the validity of the above analysis!

It should also be possible to analyse the circuit using an appropriate small-signal model, which would probably show better how use is made of the (variable) base-emitter resistance. However in using such a model I doubt it would be quite as clear how the non-linearities have been 'overlooked' in deriving the transfer function, and presumably it is these same non-linearities which contribute to the distinctive sound of the filter. However, I shall update this note with any more details as and when they arise!

July 5, 2005 postscript. Since writing the above in February 2004, another interesting paper on the Moog Ladder Filter has appeared, [4]. I have had little exposure to digital filters, but have seen the opinion expressed in several places that digital implementations of the ladder filter tend not to mirror its fabled quality, and so have wondered if it is possible somehow to 'replace' the non-linearities lost in making the linear approximation to tanh in the derivation of equations (4) and (6) above, and thus whether this would add to the sound quality of any such implementation. Without having studied much of the detail in the paper, it looks as though [4] may have come up with a method for doing this!

References

[1] R.A. Moog, *Electronic High-pass and Low-pass Filters Employing the Base to Emitter Diode Resistance of Bipolar Transistors*, USPTO No. 3,475,623, Oct 28, 1969

[2] R.A. Moog, *Voltage-Controlled Electronic Music Modules*, J. Audio Eng. Soc., vol 13 no. 3, pp200-206, July 1965

[3] R.A. Moog, *A Voltage-Controlled Low-Pass High-Pass Filter for Audio Signal Processing*, Audio Eng. Soc. preprint 413, 17$^{\text{th}}$ AES Annual Meeting, 11-15 Oct, 1965

[4] A. Huovilainen, *Non-Linear Digital Implementation of the Moog Ladder Filter*, 7$^{\text{th}}$ Int. Conf. on Digital Audio Effects (DAFx'04), Oct 5-8, 2004, Naples, Italy (`http://dafx04.na.infn.it/WebProc/Proc/P_061.pdf`)

[5] T. Stilson and J. Smith, *Analyzing the Moog VCF with Considerations for Digital Implementation*, International Computer Music Conference, Hong Kong, 1996, (`http://ccrma-www.stanford.edu/~stilti/papers/moogvcf.pdf`)

[6] T.H. Wilmshurst, *Analog Circuit Techniques with Digital Interfacing*, Newnes, 2001